Section 2.3 Differentiation

We learn:

- The definition and interpretation as slope of partial derivatives.
- How to compute partial derivatives.
- Linear approximations to a function
- The tangent plane of a function of two variables
- The gradient of a function

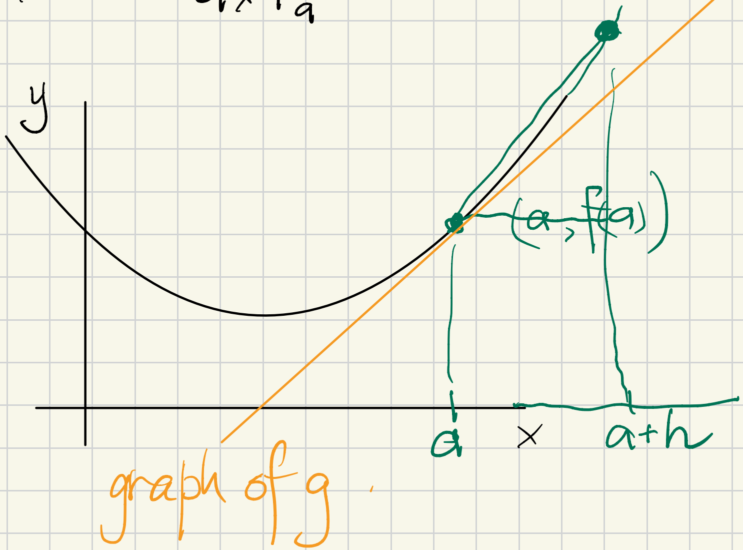
In the book there are also some theoretical things:

- what it means to be differentiable
- Theorem 9 gives a condition a function to be differentiable.

Before we get started: review of the derivative of a function $f : R \to R$.

We know the derivative of f at the point a is

$$f'(a) = \left.\frac{df}{dx}\right|_a = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$



- It represents the rate of change: how fast we are going.
- It provides a linear approximation to $f(x)$ near $x = a$.

$$g(x) = f(a) + (x-a)f'(a)$$

approximates f.

It has graph tangent to the graph of f.

Definition of partial derivatives

To make things easier we start with a function f : R^n -> R
The (partial) derivative of f(x_1, ... ,x_n) with respect to variable x_j at the point a = (a_1, ... , a_n) is

$$\lim_{h \to 0} \frac{f(a_1, \dots, a_j + h, \dots, a_n) - f(a_1, \dots, a_n)}{h}$$

$$= \left.\frac{\partial f}{\partial x_j}\right|_{\underline{x} = \underline{a}}$$

Idea: we regard all variables other than x_j as constants and differentiate as usual with respect to x_j.

Examples:
a. f(x,y) = 2x - y

$$\frac{\partial f}{\partial x} = 2 \qquad \frac{\partial f}{\partial y} = -1$$

The graph of f is the plane in R^3 that is the set of points (x, y, 2x-y). It is given by the equation z = 2x-y.
The partial derivatives are the slopes of this plane in the x and in the y directions.

b. The partial derivatives of x^3y + xy^2 at the point (1,2).

$$\frac{\partial f}{\partial x} = 3x^2y + y^2$$

Pre-class Warm-up !!

Let f(x,y) = 5xy^4 + x^2 sin(y) + y
What is ∂f/∂x?

a. 5 y^4 + 2x sin(y) ✓

b. 5 y^4 + 2x sin(y) + 1

c. 20 y^3 + 2x cos(y) + 1

d. The question doesn't mean anything, because we only know how to find the partial derivative of f at a point, and no point is given.

e. None of the above.

$$\left.\frac{\partial f}{\partial x}\right|_{(1,2)} = 5 \cdot 2^4 + 2 \cdot 1 \cdot \sin(2)$$

Linear approximation to f near a point $a \in \mathbb{R}^n$ $a = (a_1, \dots, a_n)$

g(x_1, … ,x_n) = f(a_1, … , a_n) +

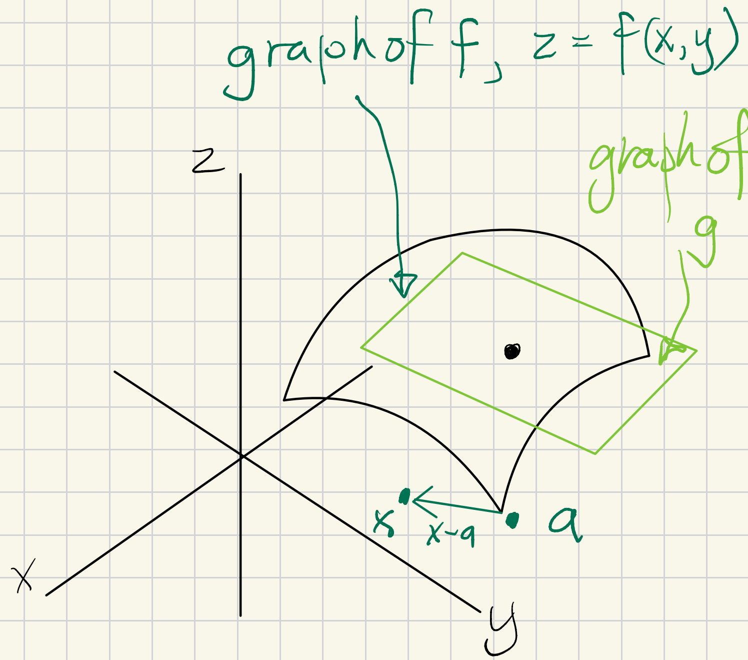
$$+ \frac{\partial f}{\partial x_1}(a)(x_1 - a_1) + \frac{\partial f}{\partial x_2}(a)(x_2 - a_2) + \cdots$$

$$+ \frac{\partial f}{\partial x_n}(a)(x_n - a_n)$$

$= z$ gives the equation of the tangent space.

new variable

The graph of the linear approximation is a linear space tangent to the graph of f at the point a.



The matrix of partial derivatives

Component
Coordinate functions:

So far we did functions f : R^n -> R.
Now we do f : R^n -> R^m
Example f(s,t) = s(1,2,3) + t^2(1,0,-1).
This f is made up of 3 functions R^2 -> R

Here $f : \mathbb{R}^2 \to \mathbb{R}^3$.

$$f(s,t) = (s+t^2,\ 2s,\ 3s-t^2)$$
$$= (f_1(s,t),\ f_2(s,t),\ f_3(s,t))$$

$f_1(s,t) = s+t^2$ $\quad f_2(s,t) = 2s$

$f_3(s,t) = 3s - t^2$

We can do $\dfrac{\partial f_3}{\partial t} = -2t$

We make a matrix of partial derivatives
where the (i,j) entry is ∂f_i / ∂x_j

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & & & \vdots \\ \vdots & & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & & \end{bmatrix}$$

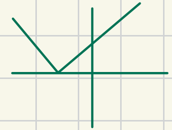
e.g.

3×2

$$\begin{bmatrix} 1 = \frac{\partial f_1}{\partial s} & \cdot \\ \cdot & \cdot \\ \frac{\partial f_3}{\partial s} = 3 & \cdot \end{bmatrix}$$

This matrix is called the derivative (matrix) of f , or the Jacobian matrix of f.

Definition of differentiability



Functions might not be differentiable everywhere. The official definition for f : R^n -> R^m to be differentiable at a point a in R^n is as follows.

Let D(a) be the derivative matrix of f at a. Then f is differentiable at a if D(a) exists and also

matrix · vector

$$\lim_{x \to a} \frac{\|f(x) - f(a) - Df(a)\cdot(x-a)\|}{\|x-a\|} = 0$$

Example $f : \mathbb{R}^n \to \mathbb{R}$

$$D = \left(\frac{\partial f}{\partial x_1} \cdots \frac{\partial f}{\partial x_n}\right)$$

$$Df(a) \cdot \begin{bmatrix} x_1 - a_1 \\ \vdots \\ x_n - a_n \end{bmatrix} = \frac{\partial f}{\partial x_1}(x_1 - a_1) + \cdots + \frac{\partial f}{\partial x_n}(x_n - a_n)$$

matrix · vector

The function
g(x) = f(a) + Df(a) (x-a)
is the best linear approximation to f near a.

Theorem 8: If f is differentiable at a then f is continuous at a.

Theorem 9: If all the partial derivatives ∂f_i / ∂x_j exist and are continuous near a then f is differentiable at a.

In the book they also define the gradient of a function f : R^n -> R but do not explain why. Their notation confuses row vectors and the column vectors used in matrix multiplication.

If $f : \mathbb{R}^n \to \mathbb{R}$

$$\operatorname{grad} f = \nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix} \text{ — a vector.}$$

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The gradient of f at a is the vector

Example. Find the gradient of
$f(x,y) = x^2y - xy^2$
at the point $(1,-1)$

What is the slope of the graph of $f(x,y) = 2x - y + 5$ in the direction of increasing x at the point $(1,3)$?

The graph is $z = 2x - y + 5$

a. 1

b. 2 $= \frac{\partial f}{\partial x}(1,3) = 2$

c. 3

Example. Let f(x,y) = 3xy^2 + x^3 + 1

a. Find the equation of the tangent plane to the graph of f at (x,y) = (1,-1).

b. Use the linear approximation of f around (x,y) = (1,-1) to approximate f(0.9,-1.1).

Solution. a. It is

$$z = \frac{\partial f}{\partial x}(1,-1) \cdot x + \frac{\partial f}{\partial y}(1,-1) y + D$$

$$= 6 \cdot x - 6y + D$$

$$f(1,-1) = 5 = 6 \cdot 1 - 6(-1) + D$$

$$D = -7$$

$$g(x,y) = 6(x-1) - 6(y+1) + 5$$

$$g(0.9,-1.1) =$$